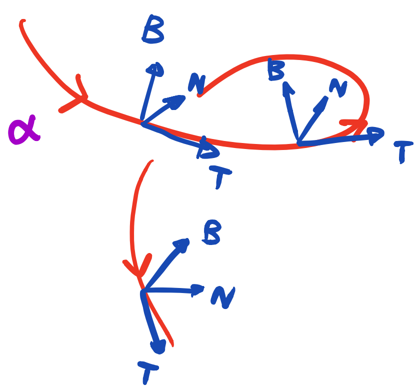
§ Space curves (do Carmo § 1.5)

Consider now a space curve

$$\alpha : I \longrightarrow \mathbb{R}^3 \quad \text{p.b.a.l.}$$



Goal: Define a "moving frame" along α whose rate of change reflects the (extrinsic) "geometry" of α

Frenet frame: $\{T, N, B\}$

Recall for plane curves: $\alpha : I \longrightarrow \mathbb{R}^2$ p.b.a.l.

Frenet frame

$$\{T, N\}, \quad T = \alpha', \quad N = JT$$

Frenet equations

$$\begin{pmatrix} T \\ N \end{pmatrix}' = \begin{pmatrix} 0 & k \\ -k & 0 \end{pmatrix} \begin{pmatrix} T \\ N \end{pmatrix}$$

$k := \langle T', N \rangle$ curvature

Note: $k = \pm |T'|$

Now, for a space curve: $\alpha : I \longrightarrow \mathbb{R}^3$ p.b.a.l.

Define: $T(s) := \alpha'(s)$ tangent

and $\boxed{k(s) := |T'(s)|}$ curvature

Note: $k \geq 0$ for space curves

$\boxed{\text{Assume: } k(s) \neq 0}$ (*)

Then, we can define:

$$N(s) := \frac{T'(s)}{|T'(s)|} \qquad \text{normal}$$

and $$B(s) := T(s) \times N(s) \qquad \text{binormal}$$

For any $\alpha : I \to \mathbb{R}^3$ p.b.a.l. satisfying (*) for all $s \in I$, we have defined smoothly along α the

Frenet frame: $\{T(s), N(s), B(s)\}$

<u>Frenet equations</u>: $\alpha : I \to \mathbb{R}^3$ p.b.a.l., $k(s) > 0 \ \forall s \in I$

$$\begin{pmatrix} T \\ N \\ B \end{pmatrix}' = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix} \qquad \cdots\cdots (\#)$$

where $\boxed{\begin{aligned} k &:= |T'| \\ \tau &:= \langle B', N \rangle \end{aligned}}$ <u>curvature</u> <u>torsion</u>

<u>Note:</u> $k \geq 0$ always, but τ can be < 0, $= 0$ or > 0.

<u>Proof of (#)</u>: Use $\{T(s), N(s), B(s)\}$ is O.N.B. $\forall s \in I$

Differentiating w.r.t. s:

$$\left.\begin{array}{lll} \langle T, T \rangle \equiv 1 & \Rightarrow & \langle T', T \rangle = 0 \\ \langle N, N \rangle \equiv 1 & \Rightarrow & \langle N', N \rangle = 0 \\ \langle B, B \rangle \equiv 1 & \Rightarrow & \langle B', B \rangle = 0 \end{array}\right\} \Rightarrow \begin{array}{l}\text{diagonal entries} \\ \text{in (\#)} = 0.\end{array}$$

By def^{n} $\quad N = \dfrac{T'}{|T'|} = \dfrac{T'}{\underset{\neq 0}{k}} \quad \Rightarrow \quad \boxed{T' = kN}$

By def^{n} $\quad B = T \times N \quad \Rightarrow \quad B' = T' \times N + T \times N'$

$$= \underbrace{kN \times N}_{= 0} + \underbrace{T \times N'}_{\perp T}$$

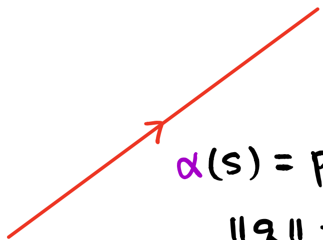
$$\Rightarrow \quad \boxed{B' = \tau N}$$

Finally,

$$\left.\begin{array}{lll} \langle N, T \rangle \equiv 0 & \Rightarrow & \langle N', T \rangle + \overbrace{\langle N, T' \rangle}^{k} = 0 \\ \langle N, B \rangle \equiv 0 & \Rightarrow & \langle N', B \rangle + \underbrace{\langle N, B' \rangle}_{\tau} = 0 \end{array}\right\} \Rightarrow \boxed{N' = -kT - \tau B}$$

$\square$

(Bad) Example 0 : Straight lines



$\alpha(s) = p + sq, \quad s \in \mathbb{R}$

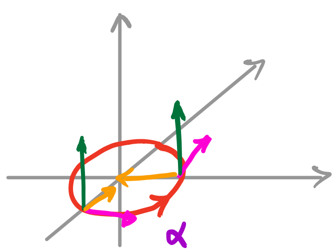
$\|q\| = 1 \quad$ p.b.a.l.

$T(s) = \alpha'(s) = q$

$T'(s) = 0$

$$\boxed{\begin{array}{l} k \equiv 0 \\ \tau \text{ not defined} \end{array}}$$

Example 1 : Circles



$$\boxed{\begin{array}{l} k \equiv \frac{1}{r} \\ \tau \equiv 0 \end{array}}$$

p.b.a.l.

$$\alpha(s) = \left(r\cos\frac{s}{r}, r\sin\frac{s}{r}\right), \quad s \in \mathbb{R}$$

$$T(s) = \alpha'(s) = \left(-\sin\frac{s}{r}, \cos\frac{s}{r}\right)$$

$$T'(s) = \frac{1}{r}\left(-\cos\frac{s}{r}, -\sin\frac{s}{r}\right)$$

$$\therefore \quad k(s) = |T'(s)| = \frac{1}{r} > 0$$

$$N(s) = \frac{T'(s)}{|T'(s)|} = \left(-\cos\frac{s}{r}, -\sin\frac{s}{r}\right)$$

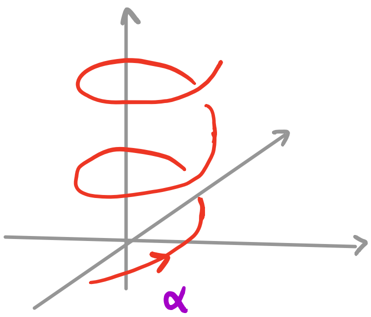
$$B(s) = T(s) \times N(s) = (0, 0, 1)$$

$$\therefore \quad \tau(s) = \langle B'(s), N(s) \rangle = 0$$

Exercise : Let $\alpha : I \to \mathbb{R}^3$, p.b.a.l., $k > 0$. Then

" α lies on a plane $P \subseteq \mathbb{R}^3$ " $\iff$ $\tau \equiv 0$.

Example 2 : Helix



p.b.a.l.

$$\alpha(s) = \left(\cos\frac{s}{\sqrt{2}}, \sin\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}}\right), \quad s \in \mathbb{R}$$

$$T(s) = \alpha'(s) = \frac{1}{\sqrt{2}}\left(-\sin\frac{s}{\sqrt{2}}, \cos\frac{s}{\sqrt{2}}, 1\right)$$

$$T'(s) = \frac{1}{2}\left(-\cos\frac{s}{\sqrt{2}}, -\sin\frac{s}{\sqrt{2}}, 0\right)$$

$$\therefore \; k(s) = |T'(s)| = \frac{1}{2}.$$

$$N(s) = \frac{T'(s)}{|T'(s)|} = \left(-\cos\frac{s}{\sqrt{2}}, -\sin\frac{s}{\sqrt{2}}, 0\right)$$

$$B(s) = T(s) \times N(s) = \frac{1}{\sqrt{2}}\left(\sin\frac{s}{\sqrt{2}}, -\cos\frac{s}{\sqrt{2}}, 1\right)$$

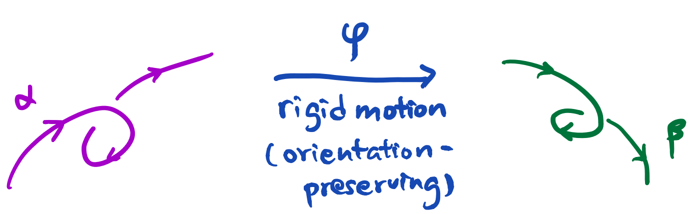
$$B'(s) = \frac{1}{2}\left(\cos\frac{s}{\sqrt{2}}, \sin\frac{s}{\sqrt{2}}, 0\right)$$

$$\therefore \; \tau(s) = \langle B'(s), N(s)\rangle = -\frac{1}{2}$$

$$\boxed{\begin{array}{l} k \equiv \frac{1}{2} \\ \tau \equiv -\frac{1}{2} \end{array}}$$

Exercise : k, τ are "geometric" quantity, i.e. they are invariant under orientation-preserving rigid motions of $\mathbb{R}^3$.

$$k_\alpha = k_\beta, \quad \tau_\alpha = \tau_\beta$$



Remark:

If $\alpha : I \to \mathbb{R}^3$ is NOT p.b.a.l. (but regular)

then "reparametrize" by $\beta = \alpha \circ \phi : J \to \mathbb{R}^3$ p.b.a.l.

define

$$\boxed{\begin{aligned} k_\alpha(t) &:= k_\beta(\phi^{-1}(t)) \\ \tau_\alpha(t) &:= \tau_\beta(\phi^{-1}(t)) \end{aligned}}$$

$\Rightarrow$ Note: Curvature is invariant under reparametrization!